AMERICAN UNIVERSITY OF BEIRUT
Faculty of Arts and Sciences
Math 202, Quiz # 2, November 27, 2010

Instructions:

1. Write your name, your ID number, and circle the number of your recitation session in the spaces provided below.
2. Solve all 5 problems. You may use for answer the back side of each page if necessary. Each problem counts for 20% of the grade.
3. Maximum time allowed is 60 minutes.
4. Use of calculators is forbidden.

Name: Key ______________________. ID #: ______________________

Recitation section: **1** 2 TH. **2** 12:30 TH. **3** 3:30 TH. **4** 11 TH.

Problem 1
Problem 2
Problem 3
Problem 4
Problem 5
Grade

1. (20 pts) Find the general solution of the following sixth order linear differential equation

$$y^{(6)} - 16y'' = x - 15e^x.$$

auxiliary equation $m^6 - 16m^2 = 0$; $m^2(m^2-4)(m^2+4) = 0$

Roots $m_1 = m_2 = 0$, $m_3 = 2$, $m_4 = -2$, $m_5 = 2i$, $m_6 = -2i$

complementary sol'tn: $y_c = c_1 + c_2 x + c_3 e^{2x} + c_4 e^{-2x} + c_5 \cos 2x + c_6 \sin 2x$.

particular solution $y = ax^3 + be^x$

$y'' = 6ax + be^x$

$y^{(6)} = be^x$

$-16(6ax + be^x) + be^x = x - 15e^x$

$-96ax - 15be^x = x - 15e^x$

$a = -\frac{1}{96}$, $b = 1$

$y_p = -\frac{1}{96}x^3 + e^x$.

general solution

$$y = -\frac{1}{96}x^3 + e^x + c_1 + c_2 x + c_3 e^{2x} + c_4 e^{-2x} + c_5 \cos 2x + c_6 \sin 2x.$$

2.(20 pts) Use, among other things, variation of parameters, to find the general solution of

$$x^2y'' - 5xy' + 8y = 8x^3.$$

First $\quad x^2y'' - 5xy' + 8y = 0$

indicial eq $\quad m(m-1) - 5m + 8 = 0 \quad m^2 - 6m + 8 = 0 \quad (m-4)(m-2) = 0$

indicial roots $\quad m_1 = 4, \; m_2 = 2$

comp. solution $\quad y_c = c_1 x^4 + c_2 x^2.$

Second $\quad y'' - \frac{5}{x}y' + \frac{8}{x^2}y = 8x.$

$y = u_1 y_1 + u_2 y_2$

system $\quad u_1' x^4 + u_2' x^2 = 0$

$u_1' \cdot 4x^3 + u_2' \cdot 2x = 8x.$

solution $\quad u_1' = \dfrac{\begin{vmatrix} 0 & x^2 \\ 8x & 2x \end{vmatrix}}{\begin{vmatrix} x^4 & x^2 \\ 4x^3 & 2x \end{vmatrix}} = \dfrac{-8x^3}{-2x^5} = \dfrac{4}{x^2}$

$u_1 = \int \frac{4}{x^2} dx = -\frac{4}{x}$

$u_2' = \dfrac{\begin{vmatrix} x^4 & 0 \\ 4x^3 & 8x \end{vmatrix}}{-2x^5} = \dfrac{8x^5}{-2x^5} = -4$

$u_2 = \int -4 \, dx = -4x$

$y_p = u_1 y_1 + u_2 y_2 = -\frac{4}{x} x^4 - 4x \, x^2$

$\therefore$

$y_p = -4x^3 - 4x^3 = -8x^3$

$y = y_c + y_p \quad 2$

$y = c_1 x^4 + c_2 x^2 - 8x^3$

3. (20 pts) Use an extended power series - method of Frobenius- to find one non-trivial solution of

$$x(x-1)y'' + (2x-1)y' + \frac{1}{4}y = 0.$$

You must determine the indicial roots, the recurrence relations for the coefficients, a formula for the n^{th} coefficient, and give the answer in series form..

$$y = \sum_{n=0}^{\infty} c_n x^{n+r}, \quad y' = \sum_0^\infty (n+r)c_n x^{n+r-1}, \quad y'' = \sum_0^\infty (n+r)(n+r-1)c_n x^{n+r-2}$$

$$\sum_0^\infty (n+r)(n+r-1)c_n x^{n+r} - \sum_0^\infty (n+r)(n+r-1)c_n x^{n+r-1}$$
$$+ \sum_0^\infty 2(n+r)c_n x^{n+r} - \sum_0^\infty (n+r)c_n x^{n+r-1} + \sum_0^\infty \frac{1}{4} c_n x^{n+r} = 0$$

$$\sum_{n=0}^\infty \left((n+r)(n+r+1)+\frac{1}{4}\right)c_n x^{n+r} - \sum_0^\infty (n+r)^2 c_n x^{n+r-1} = 0$$

$$\sum_{n=0}^\infty \left((n+r)(n+r+1)+\frac{1}{4}\right)c_n x^{n+r} - r^2 c_0 x^{r-1} - \sum_1^\infty (n+r)^2 c_n x^{n+r-1} = 0$$

$$\sum_{n=0}^\infty \left[\left((n+r)(n+r+1)+\frac{1}{4}\right)c_n - (n+r+1)^2 c_{n+1}\right] x^{n+r} - r^2 c_0 x^{r-1} = 0$$

$\therefore \quad r^2 c_0 = 0 \; ; \quad \left((n+r)(n+r+1)+\frac{1}{4}\right)c_n - (n+r+1)^2 c_{n+1} = 0$

$c_0 \neq 0 \quad \therefore \quad r_1 = r_2 = 0;$

$\therefore \quad \left(n(n+1)+\frac{1}{4}\right)c_n - (n+1)^2 c_{n+1} = 0$

$\therefore \quad \left(n+\frac{1}{2}\right)^2 c_n - (n+1)^2 c_{n+1} = 0 \qquad n = 0, 1, 2, 3, \ldots$

$\therefore \quad c_{n+1} = \dfrac{(n+\frac{1}{2})^2}{(n+1)^2} c_n \qquad n = 0, 1, 2, 3 \ldots$

or $\quad c_{n+1} = \dfrac{(2n+1)^2}{(2n+2)^2} c_n = \left(\dfrac{2n+1}{2n+2}\right)^2 c_n, \quad n = 0, 1, 2, 3.$

$\therefore \quad c_{n+1} = \left(\frac{1}{2}\right)^2\left(\frac{3}{4}\right)^2\left(\frac{5}{6}\right)^2 \cdots \left(\frac{2n+1}{2n+2}\right)^2 c_0 = \left(\dfrac{1.3.5.\cdots(2n+1)}{2.4.6.\cdots(2n+2)}\right)^2 c_0 \quad \rightarrow$

$$y = 1 + \sum_{n=1}^{\infty} \left(\frac{1.3.5.\cdots(2n-1)}{2.4.6.\cdots(2n)} \right)^2 x^n$$

4. (20 pts) Use the substitution $y = x^{1/2}z$ to find the general solution of the differential equation

$$4x^2y'' + (16x^2 + 1)y = 0$$

You are to obtain the transformed equation, write down its general solution, and then write down the solution of the original equation.Hint: $y' = \frac{1}{2}x^{-1/2}z + x^{1/2}z'$. Find y''.

$y = x^{1/2} z$

$y' = \frac{1}{2} x^{-1/2} z + x^{1/2} z'$

$y'' = -\frac{1}{4} x^{-3/2} z + x^{-1/2} z' + x^{1/2} z''$

$-x^{1/2} z + 4x^{3/2} z' + 4x^{5/2} z'' + (16x^2+1) x^{1/2} z = 0$

$-z + 4xz' + 4x^2 z'' + (16x^2+1) z = 0$

$x^2 z'' + xz' + 16x^2 z = 0$

let $4x = t$

$z' = \frac{dz}{dt} \cdot \frac{dt}{dx} = \frac{dz}{dt} \cdot 4 = 4\frac{dz}{dt}$

$z'' = \frac{dz'}{dx} = \frac{dz'}{dt} \cdot \frac{dt}{dx} = 4\frac{d^2z}{dt^2} \cdot 4 = 16\frac{d^2z}{dt^2}$

$\frac{t^2}{16} 16 \frac{d^2z}{dt^2} + \frac{t}{4} \cdot 4 \frac{dz}{dt} + t^2 z = 0$

$t^2 \frac{d^2z}{dt^2} + t\frac{dz}{dt} + t^2 z = 0$

$z = c_1 J_0(t) + c_2 Y_0(t)$

$x^{-1/2} y = c_1 J_0(4x) + c_2 Y_0(4x)$

5. (20 pts) Express $J_{-1/2}(x)$ in terms of elementary functions.

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \; x^{2n+\nu}}{2^{2n+\nu}\, n!\, \Gamma(n+\nu+1)}$$

$$J_{-1/2}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n+1-\frac{1}{2})} \left(\frac{x}{2}\right)^{2n-1/2}$$

but $\Gamma(n+\frac{1}{2}+1) = \frac{(2n+1)!}{2^{2n+1}\, n!}\sqrt{\pi}$

$$\Gamma\left(n+1-\tfrac{1}{2}\right) = \Gamma\left(1+(n-1)+\tfrac{1}{2}\right) = \frac{\left(2(n-1)+1\right)!\,\sqrt{\pi}}{2^{2(n-1)+1}(n-1)!}$$

$$\Gamma\left(n+1-\tfrac{1}{2}\right) = \frac{(2n-1)!\,\sqrt{\pi}}{2^{2n-1}(n-1)!} = \frac{(2n)\,(2n-1)!\,\sqrt{\pi}}{2n\, 2^{2n-1}\,(n-1)!}$$

$$\Gamma\left(n+1-\tfrac{1}{2}\right) = \frac{(2n)!\,\sqrt{\pi}}{2^{2n}\, n!}$$

$$J_{-\frac{1}{2}}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \; 2^{2n}\, \cancel{n!}}{\cancel{n!}\,(2n)!\,\sqrt{\pi}} \left(\frac{x}{2}\right)^{2n-1/2}$$

$$J_{-1/2}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \; 2^{2n}}{(2n)!\;\sqrt{\pi}} \; \frac{x^{2n-1/2}}{2^{2n-1/2}}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n\, \cancel{2^{2n}}\, x^{2n} \cdot x^{-1/2}}{(2n)!\,\sqrt{\pi}\;\cancel{2^{2n}} \cdot 2^{-1/2}}$$

$$= \frac{x^{-1/2}}{\sqrt{\pi}\,2^{-1/2}} \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n}}{(2n)!}$$

$$= \sqrt{\frac{2}{\pi x}} \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n}}{(2n)!} = \sqrt{\frac{2}{\pi x}}\,\cos x$$