Math 202

Quiz # 2, December 4, 2009

NAME:

ID #:

Solve all 4 problems. Use both sides of each sheet of paper if necessary. No calculators allowed. Maximum time allowed is 60 minutes.

Problem # 1:

Problem # 2:

Problem # 3:

Problem # 4:

Solution

Problem # 1. (30 pts.) Find the general solution of the following differential equation

$$y^{(6)} - y'' = 3 + e^x.$$

$y = y_c + y_p$

$y^{(6)} - y'' = 0$

$m^6 - m^2 = 0 \Rightarrow m^2(m^4 - 1) = 0$

$m^2(m^2-1)(m^2+1) = 0$

$m_1 = m_2 = 0 \quad m_3 = 1, \ m_4 = -1$

$m_5 = i \quad m_6 = -i$

$y_c = c_1 + c_2 x + c_3 e^x + c_4 e^{-x} + c_5 \cos x + c_6 \sin x$

$y_p = Ax^2 + Bxe^x$

$y_p' = 2Ax + Be^x + Bxe^x$

$y_p'' = 2A + Be^x + Be^x + Bxe^x$

$y_p'' = 2A + 2Be^x + Bxe^x$

$y_p''' = 2Be^x + Be^x + Bxe^x = 3Be^x + Bxe^x.$

$y_p^{(4)} = 3Be^x + Be^x + Bxe^x = 4Be^x + Bxe^x$

$y_p^{(5)} = 4Be^x + Be^x + Bxe^x = 5Be^x + Bxe^x$

$y_p^{(6)} = 5Be^x + Be^x + Bxe^x = 6Be^x + Bxe^x.$

$y^{(6)} - y'' = 3 + e^x$

$6Be^x + \cancel{Bxe^x} - 2A - 2Be^x - \cancel{Bxe^x} = 3 + e^x.$

$-2A + 4Be^x = 3 + e^x \Rightarrow A = -3/2 \quad B = 1/4$

$y = y_c + y_p = c_1 + c_2 x + c_3 e^x + c_4 e^{-x} + c_5 \cos x + c_6 \sin x - \frac{3}{2}x^2 + \frac{1}{4}xe^x$

Problem # 2. (25 pts.) Use the method of **variation of parameters** to find a particular solution of the following differential equation

$$x^2y'' - 4xy' + 4y = 54x^{10}.$$

This is a Cauchy Euler Equation

$y = y_c + y_p.$

The indicial equation is $m(m-1) - 4m + 4 = 0.$

$m^2 - 5m + 4 = 0 \qquad m_1 = 1 \qquad m_2 = 4 \qquad y_c = c_1 x^4 + c_2 x$

$$f(x) = \frac{54x^{10}}{x^2} = 54x^8.$$

$$W = \begin{vmatrix} x^4 & x \\ 4x^3 & 1 \end{vmatrix} = -3x^4,$$

$$u_1' = \frac{-54x^9}{-3x^4} = 18x^5 \rightarrow u_1 = \int 18x^5 dx = 3x^6$$

$$u_2' = \frac{54x^{12}}{-3x^4} = -18x^8 \rightarrow u_2 = -\int 18x^8 dx = -2x^9$$

$$y_p = u_1y_1 + u_2y_2 = 3x^{10} - 2x^{10} = x^{10}$$

$$y = y_c + y_p = c_1x^4 + c_2x + x^{10}$$

Problem # 3. (20 pts.) $J_3(x)$ is a solution of a particular second order differential equation. Write down that equation. Now answer the following question:

Prove that $x^3 J_3(x)$ is a solution of the differential equation

$$xy'' - 5y' + xy = 0, x > 0.$$

$J_3(x)$ is the solution of $x^2 y'' + xy' + (x^2 - 9)y = 0 \quad (E)$

$y = x^3 J_3 \rightarrow y' = 3x^2 J_3 + x^3 J_3'$

$y'' = 6x J_3 + 6x^2 J_3' + x^3 J_3''$

$xy'' - 5y' + xy = 6x^2 J_3 + 6x^3 J_3' + x^4 J_3'' - 15x^2 J_3 - 5x^3 J_3' + x^4 J_3$

$= x^4 J_3'' + x^3 J_3' + (x^4 - 9x^2) J_3$

$= x^2 \left[x^2 J_3'' + x J_3' + (x^2 - 9) J_3 \right)$

$= x^2 (0)$ since J_3 is the solution of the equation (E)

$= 0$

$\therefore$ $x^3 J_3(x)$ is a solution of $xy'' - 5y' + xy = 0$

Problem # 4. (25 pts.) **(a)** Use an extended power series to find one non-trivial solution y_1 of

$$y'' + \frac{1}{x}y' - 4y = 0.$$

$$y = \sum_{n=0}^{\infty} c_n x^{n+r} \qquad y' = \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2}$$

$$xy'' + y' - 4xy = 0$$

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-1} + \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1} - 4 \sum_{n=0}^{\infty} c_n x^{n+r+1} = 0$$

$$\sum_{n=0}^{\infty} (n+r)(n+r) c_n x^{n+r-1} - 4 \sum_{n=0}^{\infty} c_n x^{n+r+1} = 0$$

$$\sum_{n=-2}^{\infty} (n+r+2)^2 c_{n+2} x^{n+r+1} - 4 \sum_{n=0}^{\infty} c_n x^{n+r+1} = 0$$

$$r^2 c_0 x^{r-1} + (r+1)^2 c_1 x^r + \sum_{n=0}^{\infty} \left[(n+r+2)^2 c_{n+2} - 4c_n\right] x^{n+r+1} = 0$$

$r^2 = 0$ since $c_0 \neq 0 \Rightarrow r = 0$

$c_1 = 0 \qquad (n+r+2)^2 c_{n+2} - 4c_n = 0$

let $r = 0 \rightarrow \qquad (n+2)^2 c_{n+2} - 4c_n = 0$

$$c_{n+2} = \frac{4c_n}{(n+2)^2}$$

let $n = 0 \rightarrow c_2 = \dfrac{4c_0}{2^2}$

$n = 1 \rightarrow c_3 = \dfrac{4c_1}{3^2} = 0 = c_5 = c_7 \cdots = c_{2k+1}$

$$n=2 \longrightarrow c_4 = \frac{4c_2}{4^2} = \frac{4^2}{4^2 \cdot 2^2} c_0 = \frac{4^2}{2^4 \cdot 2^2}$$

$$n=\cancel{4} \longrightarrow c_6 = \frac{4c_4}{6^2} = \frac{4^3}{6^2 \cdot 4^2 \cdot 2^2} c_0 = \frac{4^3}{(1 \cdot 2 \cdot 3)^2 \cdot 2^6} c_0$$

$$= \frac{4^3}{(3!)^2 2^6} c_0$$

$$c_{2n} = \frac{4^n}{(n!)^2 2^{2n}} c_0 \qquad c_{2n+1} = 0$$

$$c_{2n} = \frac{\cancel{2^{2n}}}{\cancel{2^{2n}}(n!)^2} c_0 \qquad c_{2n+1} = 0$$

$$y = \sum_{n=0}^{\infty} c_n x^{n+r} = \sum_{n=0}^{\infty} c_n x^n = c_0 \sum_{n=0}^{\infty} \frac{x^{2n}}{(n!)^2}$$

(b) If y_1 is a solution of the same equation in part (a), and $y_2 = y_1 \ln x - w(x)$, is another solution, find a second order differential equation satisfied by w. Remark: This part is independent of part (a).

y_1 is a solution of the equation that is

$$y_1'' + \frac{1}{x} y_1' - 4y_1 = 0 \quad \text{or} \quad x y_1'' + y_1' - 4x y_1 = 0$$

$$y_2 = y_1 \ln x - w(x)$$

$$y_2' = y_1' \ln x + \frac{y_1}{x} - w'$$

$$y_2'' = y_1'' \ln x + \frac{y_1'}{x} + \frac{y_1' x - y_1}{x^2} - w''$$

$$= y_1'' \ln x + \frac{y_1'}{x} + \frac{y_1'}{x} - \frac{y_1}{x^2} - w''$$

y_2 is a solution of the equation

$$y_2'' + \frac{1}{x} y_2' - 4y_2 = 0$$

$$y_1'' \ln x + \frac{y_1'}{x} + \frac{y_1'}{x} - \frac{\cancel{y_1}}{x^2} - w'' + \frac{y_1'}{x} \ln x + \frac{\cancel{y_1}}{x^2} - \frac{w'}{x} - 4y_1 \ln x + 4w = 0$$

$$\ln x \underbrace{\left(y_1'' + \frac{y_1'}{x} - 4y_1\right)}_{=0} + \frac{2y_1'}{x} - w'' - \frac{w'}{x} + 4w = 0$$

$$w'' + \frac{w'}{x} - 4w = -\frac{2y_1'}{x}$$